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ALMOST AND NEAR HIGHER DERIVATION

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Abstract. In this paper we show that every near higher derivation on a Banach algebra is an almost higher derivation. Also we obtain conditions that every almost higher derivation is a near higher derivation.

1. Introduction

Let A be a Banach algebra. A linear mapping D on A is a *derivation* if

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. Also a class $\{H_0, H_1, \cdots, H_m\}$ of linear mappings on A is a *higher derivation* of rank m if

$$H_n(ab) = \sum_{i=0}^{n} H_i(a)H_{n-i}(b)$$

for $n = 1, 2, \cdots, m$ and for all $a, b \in A$ and H_0 is an identity mapping on A, that is, $H_0 = Id_A$.

We denote the space of bounded linear mappings on A by $\mathfrak{B}_K(A)$ which is bounded by a constant K. For the convenience we let

$$\begin{aligned}&\mathfrak{P}_m(\mathfrak{B}_K(A))\\ &\quad = \Big\{\{T_0, T_1, \cdots, T_m\}\Big|\quad T_0 = Id_A, T_i \in \mathfrak{B}_K(A) \quad i = 1, 2, \cdots, m\Big\}.\end{aligned}$$

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For $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ we define

$$T_n^{\vee}(a,b) = T_n(ab) - \sum_{i=0}^{n} T_i(a)T_{n-i}(b)$$

for $n = 1, 2, \cdots, m$ and for all $a, b \in A$.

The subset of $\mathfrak{P}_m(\mathfrak{B}_K(A))$ consisting of higher derivations is denoted by $\mathfrak{H}_m(\mathfrak{B}_K(A))$. That is, if $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(A))$ then $T_n^{\vee} = 0$ for all $n = 1, 2, \cdots, m$. For $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ we put

$$\begin{aligned} &d(\{T_0, T_1, \cdots, T_m\}) \\ &\quad = \inf\Big\{ \max_{1\le n\le m} ||T_n - H_n|| \Big| \{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(A)) \Big\} \end{aligned}$$

and

$$\begin{aligned} &\Big|\Big|\{T_0, T_1, \cdots, T_m\}\Big|\Big| \\ &\quad = \max_{1\le n\le m} ||T_n^{\vee}|| \\ &\quad = \max_{1\le n\le m} \sup_{||a||,||b||=1} \Big\{ \Big|\Big| T_n(ab) - \sum_{i=0}^{n} T_i(a)T_{n-i}(b) \Big|\Big| \quad \Big| \quad a, b \in A \Big\}. \end{aligned}$$

We can see that $\{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(A))$ if and only if

$$d(\{H_0, H_1, \cdots, H_m\}) = 0.$$

Note that $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ is called a $\delta-$*near higher derivation* on A of rank m if

$$d(\{T_0, T_1, \cdots, T_m\}) \le \delta$$

and also $\{T_0, T_1, \cdots, T_m\}$ is called an $\varepsilon-$*almost higher derivation* on A of rank m if

$$||\{T_0, T_1, \cdots, T_m\}|| \le \varepsilon.$$

B. E. Johnson[4] obtained conditions that every almost multiplicative map is a near multiplicative map. Note that A. M. Sinclair[6] proved that every derivation on a semisimple Banach algebra is continuous. F. Gurick[2] introduced the concept of higher derivation and N. P. Jewell[3] showed that the result of A. M. Sinclair could be extended to higher derivation. The author[5] solved the automatic continuity problem of an approximate higher derivation on a semisimple Banach algebra and investigate Hyers-Ulam stability for a higher derivation.

In this paper, we show that every near higher derivation on a Banach algebra is an almost higher derivation and obtain conditions that every almost higher derivation is a near higher derivation.

Definition 1.1. *We say that every near higher derivation on a Banach algebra A of rank m is an almost higher derivation if for each $\varepsilon \geq 0$ there exists $\delta \geq 0$ such that if $d(\{T_0, T_1, \cdots, T_m\}) \leq \delta$ for any $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ then $||\{T_0, T_1, \cdots, T_m\}|| \leq \varepsilon$.*

Theorem 1.2. *Let A be a Banach algebra. Every near higher derivation on A of rank m is an almost higher derivation.*

Proof. Let $\varepsilon \geq 0$ be given, $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ and $\delta = \frac{\varepsilon}{2K(m-2)+3}$. Also let $\{T_0, T_1, \cdots, T_m\}$ be a $\delta-$near derivation on A of rank m. Then there is a $\{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(A))$ such that for $n = 1, 2, \cdots, m$, $||T_n - H_n|| \leq \delta$. Also, for $n = 1, 2, \cdots, m$, we have

$$
\begin{aligned}
&||T_n^{\vee}(a,b)|| \\
&= \left|\left|T_n(ab) - \sum_{i=0}^{n} T_i(a)T_{n-i}(b)\right|\right| \\
&\leq ||(T_n - H_n)(ab)|| + \sum_{i=0}^{n} ||(T_i - H_i)(a)H_{n-i}(b)|| \\
&\quad + \sum_{i=0}^{n} ||T_i(a)(H_{n-i} - T_{n-i})(b)|| \\
&\leq ||(T_n - H_n)|| ||a|| ||b|| \\
&\quad + \sum_{i=1}^{n-1} ||(T_i - H_i)(a)H_{n-i}(b)|| \\
&\quad + ||(T_n - H_n)(a)b|| + ||(T_0 - H_0)(a)H_n(b)|| \\
&\quad + \sum_{i=1}^{n-1} ||T_i(a)(H_{n-i} - T_{n-i})(b)|| + ||a(T_0 - H_0)(b)|| + ||a(H_n - T_n)(b)|| \\
&\leq (3 + 2K(n-2)) \max_{1 \leq i \leq m} ||(T_i - H_i)||
\end{aligned}
$$

for all $a, b \in A$ with $||a|| = 1, ||b|| = 1$ and for all $\{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(A))$. Therefore we have

$$
\begin{aligned}
||\{T_0, T_1, \cdots, T_n\}|| &= \max_{1 \leq n \leq m} ||T_n^{\vee}|| \\
&\leq (3 + 2K(m-2)) d(\{T_0, T_1, \cdots, T_m\}) \\
&\leq (3 + 2K(m-2))\delta \\
&\leq \varepsilon.
\end{aligned}
$$

□

Definition 1.3. *We say that every almost higher derivation on a Banach algebra A of rank m is a near higher derivation if for each $\varepsilon \geq 0$ there exists $\delta \geq 0$ such that if $||\{T_0, T_1, \cdots, T_m\}|| \leq \delta$ for any $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ then $d(\{T_0, T_1, \cdots, T_m\}) \leq \varepsilon$.*

Example 1.4. *Let A be a Banach algebra of 2×2 matrices which the matrix is of the form*

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$$

for each scalar a and b. Let $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ be given. Define mappings $\{H_0, H_1, H_2\}$ by

$$\begin{aligned} H_0 &= Id, \\ H_1\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ \varepsilon_1 b & 0 \end{pmatrix}, \\ H_2\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ \varepsilon_2 b & 0 \end{pmatrix}. \end{aligned}$$

Then $\{H_0, H_1, H_2\}$ is a higher derivation of rank 2. Now define mappings $\{T_0, T_1, T_2\}$ by

$$\begin{aligned} T_0 &= Id, \\ T_1\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} &= \begin{pmatrix} \varepsilon_1 a & 0 \\ 0 & \varepsilon_1 a \end{pmatrix}, \\ T_2\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} &= \begin{pmatrix} \varepsilon_2 a & 0 \\ 0 & \varepsilon_2 a \end{pmatrix}. \end{aligned}$$

Then $||T_1^{\vee}|| \leq \varepsilon_1, \quad ||T_2^{\vee}|| \leq \varepsilon_2 \quad ||T_1 - H_1|| \leq \varepsilon_1$ and $||T_2 - H_2|| \leq \varepsilon_2$ Therefore $\{T_0, T_1, T_2\}$ is $max\{\varepsilon_1, \varepsilon_2\}-$almost higher derivation of rank 2 and also it is a $max\{\varepsilon_1, \varepsilon_2\}-$near higher derivation.

Theorem 1.5. *Let A be a finite dimensional Banach algebra. Then every almost higher derivation on A of any rank m is a near higher derivation.*

Proof. Note that the dimension of $(\mathfrak{B}_K(A))^m$ is also finite. Let $\varepsilon \geq 0$ be given and

$$\begin{aligned} \mathfrak{C} &= \Big\{\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A)) \Big| d(\{T_0, T_1, \cdots, T_m\}) \geq \varepsilon\Big\}, \\ \mathfrak{G}_\delta &= \Big\{\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A)) \Big| ||\{T_0, T_1, \cdots, T_m\}|| > \delta\Big\}. \end{aligned}$$

We claim that $\mathfrak{G}_\delta \subseteq (\mathfrak{B}_K(A))^m$ is open and $\mathfrak{C} \subseteq (\mathfrak{B}_K(A))^m$ is closed.

Let $\varepsilon \geq 0$ be given and there exists

$$\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,m)}\} \in (\mathfrak{B}_K(A))^m \backslash \mathfrak{G}_\delta$$

such that

$$\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,m)}\} \to \{J_0, J_1, \cdots, J_m\}$$

in $\mathfrak{G}_\delta^c$ as $t \to \infty$. Since $\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,m)}\} \notin \mathfrak{G}_\delta$ for every (t, n),

$$\sup_{||a||,||b||=1} \Big|\Big| J_{(t,n)}(ab) - \sum_{i=0}^{m} J_{(t,i)}(a) J_{(t,n-1)}(b) \Big|\Big| \leq \delta$$

and there is a constant N such that for all $t \geq N$ and $n = 1, 2, \cdots, m$

$$||J_n - J_{(t,n)}|| \leq \varepsilon.$$

Then for each n we have

$$\begin{aligned}
&||J_n^{\vee}|| \\
&\quad = \sup_{||a||,||b||=0} \Big|\Big| J_n(ab) - \sum_{i=0}^{n} J_i(a) J_{n-i}(b) \Big|\Big| \\
&\quad = \sup_{||a||,||b||=0} \Big(||(J_n - J_{(t,n)})(ab)|| + ||J_{(t,n)}(ab) \\
&\qquad - \sum_{i=0}^{n} J_{(t,i)}(a) J_{(t,n-i)}(b)|| \\
&\qquad + \sum_{i=0}^{n} ||J_{(t,i)}(a) J_{(t,n-i)}(b) - J_i(a) J_{n-i}(b)|| \Big) \\
&\quad \leq \varepsilon + \delta + \sup_{||a||,||b||=0} \sum_{i=0}^{n} ||(J_{(t,i)} - J_i)(a) J_{(t,n-i)}(b)|| \\
&\qquad + \sup_{||a||,||b||=0} \sum_{i=0}^{n} ||J_i(a)(J_{(t,n-i)} - J_{n-i})(b)||
\end{aligned}$$

$$\leq \varepsilon + \delta + \sup_{||a||,||b||=0} \sum_{i=1}^{n-1} ||(J_{(t,i)} - J_i)(a) J_{(t,n-i)}(b)||$$
$$+ ||(J_{(t,n)} - J_n)(a) J_{(t,0)}(b)||$$
$$+ \sup_{||a||,||b||=0} \sum_{i=1}^{n-1} ||J_i(a)(J_{(t,n-i)} - J_i)(b)|| + ||J_0(a)(J_{(t,n)} - J_n)(b)||$$
$$\leq \varepsilon + \delta + 2\varepsilon + 2K(n-2)\varepsilon$$
$$= M\varepsilon + \delta, \quad (where \quad M = 2K(n-2)+3).$$

Since ε was arbitrary, $||J_n^{\vee}|| \leq \delta$ for $1 \leq n \leq m$ and so

$$||\{J_0, J_1, \cdots, J_n\}|| \leq \delta.$$

This states that $\{J_0, J_1, \cdots, J_n\} \notin \mathfrak{G}_\delta$ and thus $\{J_0, J_1, \cdots, J_n\} \in \mathfrak{G}_\delta^c$. This says that $\mathfrak{G}_\delta$ is open.

To show that $\mathfrak{C}$ is closed, we let

$$\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\} \to \{J_0, J_1, \cdots, J_m\}$$

in $\mathfrak{C}$ as $t \to \infty$. Then

$$||\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\}||$$
$$= \inf \Big\{ \max_{1 \leq n \leq m} ||H_n - J_{(t,n)}|| \Big| \{H_0, H_1, \cdots, H_n\} \in \mathfrak{H}_m(\mathfrak{B}_K(A)) \Big\}$$
$$\geq \varepsilon$$

and for arbitrary $\varepsilon' \geq 0$ there is N such that for all n $(n = 1, 2, \cdots, m)$, and $t > N$

$$||J_n - J_{(t,n)}|| < \varepsilon'.$$

Thus for every $(n = 1, 2, \cdots, m)$, we have

$$||H_n - J_n|| \geq ||H_n - J_{(n,t)}|| - ||J_{(n,t)} - J_n|| \geq \varepsilon - \varepsilon'.$$

Since ε' was arbitrary, $d(\{J_0, J_1, \cdots, J_n\}) \geq \varepsilon$. Therefore $\{J_0, J_1, \cdots, J_n\} \in \mathfrak{C}$ and thus $\mathfrak{C}$ is closed.

Since $\mathfrak{C} \subseteq \mathfrak{B}_K(A)^m$ and $dim(\mathfrak{B}_K(A)^m) < \infty$, $\mathfrak{C}$ is compact. Since

$$\mathfrak{C} \subseteq \mathfrak{B}_K(A)^m \backslash \mathfrak{H}_m(\mathfrak{B}_K(A)) \subseteq \bigcup_{\delta > 0} \mathfrak{G}_\delta,$$

there exist $\delta_1, \delta_2, \cdots, \delta_l \geq 0$ such that

$$\bigcup_{i=1}^{l} \mathfrak{G}_{\delta_i} \supset \mathfrak{C}.$$

Let $\triangle = \min\{\delta_1, \delta_2, \cdots, \delta_l\}$. Then $C \subset \mathfrak{G}_\triangle$. Now let $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ with $||\{T_0, T_1, \cdots, T_m\}|| \le \triangle$. Then $\{T_0, T_1, \cdots, T_m\} \notin \mathfrak{C}$. Therefore

$$d(\{T_0, T_1, \cdots, T_m\}) < \varepsilon.$$

This says that every almost higher derivation on A of rank m is a near higher derivation. □

Theorem 1.6. *Let A be a Banach algebra. Every almost higher derivation on A of any rank m is a near higher derivation if and only if for any*

$$\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$$

with $J^{\vee}_{(t,n)} \to 0$ as $t \to \infty$ $(n = 1, 2, \cdots, m)$, there exists

$$\{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(B_K(A))$$

such that for each n $(n = 1, 2, \cdots, m)$, $J_{(t,n)} - H_n \to 0$ as $t \to \infty$.

Proof. Let $\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$ and $J^{\vee}_{(t,n)} \to 0$ as $t \to \infty$ $(n = 1, 2, \cdots, m)$. Then for any $\varepsilon \ge 0$ we can choose δ and t such that

$$||\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\}|| = \max_{1 \le i \le m} ||J^{\vee}_{(t,i)}|| \le \delta.$$

By hypothesis,

$$\begin{aligned} &d(\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\}) \\ &\quad = \inf\left\{ \max_{1 \le n \le m} ||J_{(t,n)} - H_n|| \,\middle|\, \{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_k(A)) \right\} \\ &\quad \le \varepsilon. \end{aligned}$$

Thus there exists $\{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(A))$ such that $J_{(t,n)} - H_n \to 0$ as $t \to \infty$.

Conversely, suppose that $\varepsilon \ge 0$ and

$$\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\} \in \mathfrak{P}_m(\mathfrak{B}_K(A))$$

and $J^{\vee}_{(t,n)} \to 0$ as $t \to \infty$ $\quad (n = 1, 2, \cdots, m)$. Then there exists $\delta \ge 0, N$ and

$$\{H_0, H_1, \cdots, H_n\} \in \mathfrak{H}_m(\mathfrak{B}_k(A))$$

such that for every $t \ge N$ and $n = 1, 2, \cdots, m$, $||J^{\vee}_{(t,n)}|| \le \delta$ implies $||J_{(t,n)} - H_n|| \le \varepsilon$. That is, if

$$||\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\}|| \le \delta$$

then

$$d(\{J_{(t,0)}, J_{(t,1)}, \cdots, J_{(t,n)}\}) \leq \varepsilon.$$

Thus every almost higher derivation on A of any rank m is a near higher derivation. □

EXAMPLE 1.7. Let $C^\infty[0,1]$ denote the algebra of all complex valued functions on [0,1] which have infinitely differentiable functions. J. F. Feinstein[1] got several Banach algebra norms on $C^\infty[0,1]$. We give a Banach algebra norm on $C^\infty[0,1]$ by

$$||f|| = \sum_{n=0}^{\infty} \frac{||f^{(n)}||_\infty}{(n!)^2}$$

where $||f||_\infty = \sup_{t\in[0,1]} |f(t)|$. Then $||f'|| \leq ||f||$ and $||f^{(k)}|| \leq (k!)^2||f||$, because for any $k \geq 1$

$$\frac{1}{(k!)^2}\Big(\frac{1}{(2!)^2} + \frac{(2!)^2}{(3!)^2} + \frac{(3!)^2}{(4!)^2} + \cdots\Big)||f^{(k)}|| \leq \frac{1}{(k!)^2}||f^{(k)}|| \leq ||f||.$$

We denote this Banach algebra by $C^\infty([0,1],(n!)^2)$.

Define $\{H_0, H_1, \cdots, H_m\} \in \mathfrak{H}_m(\mathfrak{B}_K(C^\infty([0,1],(n!)^2)))$ by $H_0 = Id$ and $H_n(f) = f^{(n)}$ for $n = 1, 2, \cdots, m$. Also define $\{T_0, T_1, \cdots, T_m\} \in \mathfrak{P}_m(\mathfrak{B}_K(C^\infty([0,1],(n!)^2)))$ by $T_0 = Id$ and $T_n(f) = f^{(n)} + \varepsilon f$ for $n = 1, 2, \cdots, m$. Then $||T_n(f) - H_n(f)|| \leq \varepsilon||f||$ and

$$\begin{aligned} ||T_n^\vee(fg)|| &= ||T_n(fg) - \sum_{i=0}^{m} T_i(f)T_{n-i}(g)|| \\ &= ||(fg)^{(n)} + \varepsilon fg - \sum_{i=0}^{m}(f^{(i)} + \varepsilon f)(g^{(n-i)} + \varepsilon g)|| \\ &\leq \varepsilon\Big(\varepsilon + \sum_{i=1}^{n-1}(i!)^2((m-i)!)^2\Big)||f||||g|| \end{aligned}$$

for $n = 1, 2, \cdots, m$ and $f, g \in C^\infty[0,1]$. Therefore $\{T_0, T_1, \cdots, T_m\}$ is an almost and near higher derivation.

DEFINITION 1.8. *Let A be a commutative Banach algebra. A linear mapping D on A is a ρ-derivation (or module derivation) if*

$$D(ab) = \rho(b)D(a) + \rho(a)D(b)$$

for all $a, b \in A$, which ρ is a continuous homomorhpism on A . Also a class $\{H_0 = \rho, H_1, \cdots, H_m\}$ of linear mappings on A is a ρ-higher *derivation of rank m if*

$$H_n(ab) = \sum_{i=0}^{n} H_i(a)H_{n-i}(b)$$

for $n = 1, 2, \cdots, m$ and for all $a, b \in A$. As well as almost and near higher derivations, we can define almost and near ρ-higher derivations.

We can see that all theorems above hold for the case of ρ-higher derivation.

THEOREM 1.9. *Let $\{T_0 = \rho, T_1, T_2\} \in \mathfrak{P}_2(\mathfrak{B}_K(C^\infty([0,1],(n!)^2)))$ be an $\varepsilon-$almost ρ-higher derivation of rank 2 with $\rho(z^i)T_n(z^j) = 0$ for $i + j > s$ (for some integer s and $i, j = 0, 1, \cdots, n$). Then $\{T_0, T_1, T_2\}$ is a near ρ-higher derivation.*

Proof. For the convenience, we denote $\rho(f)g$ by fg. We prove the following formula by induction; For $m \geq 2$, we have

$$\begin{aligned} T_1(z^l) =& T_1^\vee(z, z^{l-1}) + zT_1^\vee(z, z^{l-2}) + z^2T_1^\vee(z, z^{l-3}) \\ &+ \cdots + z^{l-2}T_1^\vee(z,z) + lz^{l-1}T_1(z). \end{aligned}$$

It is trivial for $l = 2$. Assume that it holds for l. Then

$$\begin{aligned} T_1(z^{l+1}) =& T_1^\vee(z, z^l) + zT_1(z^l) + z^lT_1(z) \\ =& T_1^\vee(z, z^l) + zT_1^\vee(z, z^{l-1}) \\ &+ \cdots + z^{l-1}T_1^\vee(z,z) + (l+1)z^lT_1(z). \end{aligned}$$

For any polynomial $p(z) = a_0 + a_1z + \cdots + a_tz^t \quad (t \geq s)$, we have

$$\begin{aligned} T_1(p) =& T_1(a_0) + a_1T_1(z) + a_22zT_1(z) + \cdots + a_ttz^{t-1}T_1(z) \\ &+ T_1^\vee(z,z)(a_2 + a_3z + \cdots + a_tz^{t-2}) \\ &+ T_1^\vee(z,z^2)(a_3 + a_4z + \cdots + a_tz^{t-3}) \\ &+ \cdots + T_1^\vee(z, z^{t-1}(a_t + T_1^\vee(z, z^t). \end{aligned}$$

Since $z^iT_n(z^j) = 0$ for $i + j > s$ for some integer s and $i, j = 0, 1, 2$ it is easy to show that $z^iT_1^\vee(z, z^j) = 0$ for $i + j > s$. Thus

$$\begin{aligned} T_1(p) =& p'T_1(z) + T_1(a_0) + T_1^\vee(z,z)(a_2 + a_3z + \cdots + a_{s-3}z^{s-1}) \\ &+ T_1^\vee(z,z^2)(a_3 + a_4z + \cdots + a_{s-4}z^{s-2}) + \cdots + T_1^\vee(z,z^s)(a_{s+1}). \end{aligned}$$

Since $n!|a_n| \leq ||p^{(n)}||_\infty \leq ||p^{(n)}|| \leq (n!)^2||p||$, $|a_n| \leq (n!)||p||$ for all $n = 0, 1, 2, \cdots$. Thus we have

$$||T_1(p) - p'T_1(z)|| \leq \varepsilon s(1 + 2! + 3! + \cdots + (s-3)!)||p|| = \varepsilon'||p||$$

for $\varepsilon' = m(1 + 2! + 3! + \cdots + (s-3)!)$.

We define $H_0, H_1 \in \mathfrak{B}_K(C^\infty([0,1],(n!)^2))$ by $H_0 = \rho, H_1(f) = f'T_1(z)$. Then H_1 is a derivation and for all $f \in C^\infty([0,1],(n!)^2)$

$$||T_1(f) - H_1(f)|| \leq \varepsilon' ||f||.$$

By induction, we have

$$\begin{aligned} T_2(z^l) =& T_2^\vee(z, z^{l-1}) + zT_2^\vee(z, z^{l-2}) + z^2T_1^\vee(z, z^{l-3}) + \cdots + z^{l-2}T_2^\vee(z,z) \\ &+ T_1(z)(T_1^\vee(z, z^{l-2}) + 2zT_1^\vee(z, z^{l-3}) + \cdots + (l-2)z^{l-3}T_1^\vee(z,z)) \\ &+ \frac{l(l-1)}{2}z^{l-2}T_1(z)^2 + lz^{l-1}T_2(z) \end{aligned}$$

for $l \geq 2$. It is trivial for $n = 2$. Assume that it is true for l. Then

$$\begin{aligned} T_2(z^l) =& T_2^\vee(z, z^l) + zT_2(z^l) + z^lT_2(z) + T_1(z)T_1(z^l) \\ =& T_2^\vee(z, z^l) + zT_2^\vee(z, z^{l-1}) + \cdots + z^{l-1}T_2^\vee(z,z) \\ &+ T_1(z)(T_1^\vee(z, z^l) + 2zT_1^\vee(z, z^{l-2}) \\ &+ \cdots + (l+1)z^{l-2}T_1^\vee(z,z)) \\ &+ \frac{l(l+1)}{2}z^{l-1}T_1(z)^2 + (l+1)z^lT_2(z). \end{aligned}$$

For any polynomial $p(z) = a_0 + a_1z + \cdots + a_tz^t \quad (t \geq s)$,

$$\begin{aligned} T_2(p) =& T_2(a_0) + a_1T_2(z) + a_2 2zT_2(z) \cdots + a_t tz^{t-1}T_2(z) \\ &+ \frac{T_1(z)^2}{2}(a_2 2 + a_3 3 \cdot 2z \cdots + a_t t(t-1)z^{t-2}) \\ &+ T_2^\vee(z,z)(a_2 + a_3z + \cdots + a_tz^{t-2}) \\ &+ T_2^\vee(z,z^2)(a_3 + a_4z + \cdots + a_tz^{t-3}) \\ &+ \cdots + T_2^\vee(z, z^{t-1})a_t \\ &+ T_1(z)T_1^\vee(z,z)(a_3 + a_4 2z + \cdots + a_t(t-2)z^{t-3}) \\ &+ T_1(z)T_1^\vee(z,z^2)(a_4 + a_5 2z + \cdots + a_t(t-3)z^{t-4}) \\ &+ \cdots + T_1(z)T_1^\vee(z, z^{t-2})a_t. \end{aligned}$$

Thus we have

$$
\begin{aligned}
T_2(p) =& p'T_2(z) + \frac{1}{2}p''T_2(z) + T_2(a_0) \\
&+ T_2^{\vee}(z,z)(a_2 + a_3 z + \cdots + a_{s+1}z^{s-1}) \\
&+ T_2^{\vee}(z,z^2)(a_3 + a_4 z + \cdots + a_{s+1}z^{s-2}) \\
&+ \cdots + T_2^{\vee}(z,z^s)a_{s+1} \\
&+ T_1(z)T_1^{\vee}(z,z)(a_3 + a_4 2z + \cdots + a_{s+2}sz^{s-1}) \\
&+ T_1(z)T_1^{\vee}(z,z^2)(a_4 + a_5 2z + \cdots + a_{s+2}(s-1)z^{s-2}) \\
&+ \cdots + T_1(z)T_1^{\vee}(z,z^s)a_{s+2}.
\end{aligned}
$$

By the calculation above we get

$$
\begin{aligned}
&||T_2(p) - p'T_2(z) - \frac{1}{2}f''T_1(z)^2|| \\
&\quad \le \varepsilon s(1 + 2! + 3! + \cdots + (s+1)!)||p|| \\
&\quad + \varepsilon s||T_1||(3! + 4! + \cdots + (s+2)!)||p|| \\
&\quad := \varepsilon''||p||
\end{aligned}
$$

for a constant ε'', where $\varepsilon'' \to 0$ as $\varepsilon' \to 0$.

We define $H_2 \in \mathfrak{B}_K(C^\infty([0,1],(n!)^2))$ by $H_2(f) = f'T_2(z) + \frac{1}{2}f''T_1(z)^2$. Then $\{\rho, H_1, H_2\}$ is a ρ-higher derivation of rank 2 and for all $f \in C^\infty([0,1],(n!)^2)$

$$||T_2(f) - H_2(f)|| \le \varepsilon''||f||.$$

□

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